B.Sc Part III (H)
Paper - V

State and prove More - osgood theorem.

Statement:— Let the double limit $\lim_{\substack{x\to a\\ y\to b}} f(x,y)$ exist and be equal to l and let the limit $\lim_{x\to a} f(x,y)$ exist for each constant value of y in the nbd. of $y=b$ and likewise let the limit $\lim_{y\to b} f(x,y)$ exist for each constant value of x in the nbd. of $x=a$. Then.

$$\lim_{x\to a}\lim_{y\to b} f(x,y) = \lim_{y\to b}\lim_{x\to a} f(x,y) = l.$$

Proof:— Since the limit $\lim_{x\to a} f(x,y)$ exists for each value of y in the nbd. of $y=b$, we shall obtain an aggregate of these limiting values which defines a function of y, say $F(a,y)$. Thus we have.

$$\lim_{x\to a} f(x,y) = F(a,y) \qquad (1)$$

Where $F(a,y)$ may or may not be identical with $f(a,y)$.

Let $\epsilon > 0$ be given

Since $\lim_{x\to a} f(x,y) = F(a,y)$, therefore there exists $\delta_1 > 0$ such that for each value of y in the nbd. of $y=b$ i.e. for $|y-b| < \delta_1$, we have

$$|F(a,y) - f(x,y)| < \epsilon/2 \qquad (2)$$

for all x satisfying $|x-a| < \delta_2$

Also, from the existence of double limit at (a,b), there exists $\delta_2 > 0$ such that

$$|f(x,y) - l| < \epsilon/2 \qquad (3)$$

for all x,y satisfying $|x-a| < \delta_2$, $|y-b| < \delta_2$

Let $\delta = \min.(\delta_1, \delta_2)$. Then we have

$$|F(a,y) - l| = |F(a,y) - f(x,y) + f(x,y) - l|$$
$$< |F(a,y) - f(x,y) + f(x,y) - l|$$
$$< \epsilon/2 + \epsilon/2 \text{ by virtue of (2) and (3).}$$

It follows, therefore that $\lim_{y\to b} F(a,y) = l$.

$$\Rightarrow \underset{y\to b}{Lt}\ \underset{x\to a}{Lt}\ f(x,y) = l \text{ by (1)}$$

Similarly it can be shown that

$$\underset{x\to a}{Lt}\ \underset{y\to b}{Lt}\ f(x,y) = l.$$

Thus $\underset{\substack{x\to a\\ y\to b}}{Lt} f(x,y) = \underset{x\to a}{Lt}\ \underset{y\to b}{Lt}\ f(x,y)$

$$= \underset{y\to b}{Lt}\ \underset{x\to a}{Lt}\ f(x,y) = l.$$

It has to be noted that the condition given in the theorem is only a sufficient but not a necessary condition for the interchange of the order of repeated limits. That is, if the two repeated limits $\underset{x\to a}{Lt}\ \underset{y\to b}{Lt}\ f(x,y)$ and $\underset{y\to b}{Lt}\ \underset{x\to a}{Lt}\ f(x,y)$ are equal, then the double limit $\underset{\substack{x\to a\\ y\to b}}{Lt} f(x,y)$ may or may not exist. But if the two repeated limits are unequal, then the double limit does not exist.

Example: ~~Show that~~ Let $f(x,y) = \dfrac{xy}{x^2+y^2}$

Show that the repeated limits exist at $(0,0)$ and are equal, but the double limit does not exist.

Solution:

$$\underset{y\to 0}{Lt}\ \underset{x\to 0}{Lt}\ \frac{xy}{x^2+y^2} = \underset{y\to 0}{Lt}\ \frac{0}{0+y^2} = 0$$

$$\underset{x\to 0}{Lt}\ \underset{y\to 0}{Lt}\ \frac{xy}{x^2+y^2} = \frac{0}{x^2+0} = 0.$$

Thus the two repeated limits are equal.

Now, we evaluate the double limit $\underset{\substack{x\to 0\\ y\to 0}}{Lt} \dfrac{xy}{x^2+y^2}$

By putting $y=x$, we have $\underset{x\to 0}{Lt}\ \dfrac{x^2}{x^2+y^2} = \dfrac{1}{2}$

Thus we see that though the repeated limits exist and are equal, yet the double limit does not exist.